Deconfinement and Percolation in SU(2) Gauge Theory

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Abstract:

Effective theories are helpful tools to gain an intuitive insight into phenomena governed by complex laws. In this work we show by means of Monte Carlo simulations that $Z(2)$ spin models with only spin-spin interactions approximate rather well the critical behavior of $SU(2)$ gauge theory at finite temperature. These models are then used to formulate an effective description of Polyakov loop percolation, which is shown to reproduce the thermal critical behaviour of $SU(2)$ gauge theory.

1 Introduction

The global $Z(N)$ center symmetry plays a key role in the confinement-deconfinement transition of $SU(N)$ gauge theories. The spontaneous breaking of this symmetry is responsible for the phase transition, and the corresponding order parameter is the Polyakov loop [1, 2]. In full QCD the dynamical quarks act as a (small) external magnetic field [3] and the conventional formalism of spontaneous symmetry breaking does not apply. Therefore it seems helpful to consider an alternative approach to critical behavior of $SU(N)$ gauge theory, which may be more readily generalizable to full QCD.

In [4] it was suggested that the deconfinement phase transition in $SU(N)$ gauge theory can be viewed as a percolation transition of suitably defined Polyakov loop clusters. In [5, 6] such an approach has been applied to (2+1)- and (3+1)-dimensional $SU(2)$ pure gauge theory; the clusters are formed by joining nearest-neighbour like-sign Polyakov loops with a bond probability similar to the one used to define the "physical droplets" in the Ising model [7]. In [5, 6], the bond probability was determined from an effective theory of Polyakov loops, which was derived by a strong coupling expansion [3]. Hence the validity of this approach is limited to the strong coupling limit of $SU(2)$, and the calculations were thus done on lattices with a temporal extent of only two lattice spacings. It is thus not clear how to formulate a percolation picture for $SU(2)$ in the continuum limit.

The aim of the present paper is to clarify whether the deconfinement phase transition in (3+1)-dimensional $SU(2)$ gauge theory can be described more generally as a percolation transition. We will derive an effective theory for $Z(2)$ spin variables containing up to 19 spin-spin couplings which are obtained by solving Schwinger-Dyson equations, as originally proposed in [8, 9]. This method is independent of the number N_τ of lattice spacings in the time direction and we will apply it to two different cases, $N_\tau = 2$ and 4. The particular effective theory studied here is then shown to allow an equivalent perco-

lation formulation of the deconfinement transition. The corresponding clusters are, with good approximation, the "physical droplets" of $SU(2)$.

2 Percolation and the Effective Theory

We are interested in constructing an effective theory of Z(2) spin variables $\{s_{\mathbf{n}} = \pm 1\}$, defined as the signs of Polyakov loops at the spatial sites $\mathbf{n}$. The effective Hamiltonian $\mathcal{H}(s,\beta)$ of the signs $\{s_{\mathbf{n}}\}$ of the Polyakov loop configurations can be defined through the equation [8, 9]

$$\exp[\mathcal{H}(s,\beta)] \;=\; \int [dU] \prod_{\mathbf{n}} \delta[s_{\mathbf{n}}, sgn(L_{\mathbf{n}})] \, \exp(-S_W(\beta)), \tag{1}$$

where $L_{\mathbf{n}}$ is the value of the Polyakov loop at the spatial point $\mathbf{n}$ and S_W the Wilson lattice action which depends on the gauge coupling β. Eq. (1) shows that all degrees of freedom of the original $SU(2)$ field configurations are integrated out, leaving only the distribution of the corresponding Ising-projected configuration.

The problem is now how to determine the explicit form $\mathcal{H}(s,\beta)$, starting from the original Polyakov loop configurations. In general, $\mathcal{H}(s,\beta)$ will contain infinitely many operators. However, in the vicinity of the critical point where the correlations become long ranged one expects that $\mathcal{H}(s,\beta)$ can be well approximated by a finite sum of different terms [2]. So we write

$$\mathcal{H}(s,\beta) \;=\; \sum_{\mathbf{n}} \sum_{\alpha} \kappa_\alpha(\beta) O^\alpha_{\mathbf{n}}, \tag{2}$$

where $\kappa_\alpha(\beta)$ are the couplings, $O^\alpha_{\mathbf{n}}$ are the spin-spin operators at site $\mathbf{n}$ and α specifies the specific spin-spin interaction; e.g., in the simplest case there would be only a single term $\kappa_1 O^1_{\mathbf{n}} = \kappa_1 \sum_{\hat{\mu}=1}^{4} s_{\mathbf{n}} s_{\mathbf{n}+\hat{\mu}}$, with $\hat{\mu}$ denoting a unit vector in μ-direction. In contrast to [9], we here do not include multi-spin operators in Eq. (2), because it is not known how to define clusters in this case.

To calculate the couplings, we use the following set of Schwinger-Dyson equations

$$\langle O^\gamma_{\mathbf{n}} \rangle \;=\; -\langle O^\gamma_{\mathbf{n}} \, \exp(-2 \sum_{\alpha} \kappa_\alpha O^\alpha_{\mathbf{n}}) \rangle, \tag{3}$$

which are derived by exploiting the $Z(2)$ symmetry of $\mathcal{H}(s,\beta)$ [8]. Here $< ... >$ denotes averages taken in the (3+1)-dimensional $SU(2)$ gauge theory. Eqs. (3) establish a relation between thermal averages of the operators $O^\gamma_{\mathbf{n}}$ and the couplings κ_γ. These equations are, however, implicit in the couplings. They can be solved by means of the Newton method, based on successive approximations [9]. One starts by making a guess about the values of the couplings and derives from Eqs. (3) another set of values for κ_γ's. After a sufficient number of iterations, the series of partial values for the κ_γ's will converge to the solution of Eqs. (3).

We notice that the general set of equations (3) refers to a single point $\mathbf{n}$ of the spatial volume. However, the thermal averages are independent of the particular point $\mathbf{n}$, so it doesn't matter where we decide to take the averages. We thus explore translational invariance and the Eqs. (3) should thus be understood as being averaged over the entire lattice. This reduces considerably the effect of thermal fluctuations and, consequently, the errors on the final κ_γ's.

With this we have all necessary tools to derive an effective theory for $SU(2)$ from the Polyakov loop configurations.

In general, the approximation improves the more operators we include in (2). The fact that one must restrict the choice to some subset of operators involves a truncation error in addition to the statistical one. The error due to truncation is difficult to control. It can, however, be estimated by calculating the parameters of the effective theory for different sets of operators. We also need to establish a criterium to judge how well the effective theory approximates the original one. A good option could be to compare average values we get from the configurations produced by simulating the effective theory with the corresponding quantities measured on the original Ising-projected Polyakov loop configurations. In particular, we used the lattice average of the magnetization m,

$$m \;=\; \frac{1}{V}|\sum_i s_i|, \tag{4}$$

(V is the spatial lattice volume) for such a test.

We point out that the approach we have described can be used for any value of the number N_τ of lattice spacings in the temperature direction, although we have applied the method here only to $SU(2)$ in $(3+1)$ dimensions for $N_\tau = 2$ and 4.

The magnetization transition in the Ising model can be described as percolation transition of appropriately defined site-bond clusters [7]. The clusters are defined by the condition that two nearest neighboring spins of the same sign belong to a cluster with *bond probability* (or bond weight)

$$p = 1 - \exp(-2\kappa). \tag{5}$$

Here κ is the Ising coupling divided by the temperature.

It was recently proved that this result can be extended to other spin models [11, 10, 12]. In particular, if a theory contains only ferromagnetic spin-spin interactions, the above percolation picture can be trivially extended by introducing a bond between each pair of interacting spins, and a relative bond probability

$$p_\alpha = 1 - \exp(-2\kappa_\alpha), \tag{6}$$

where κ_α is the coupling associated to the α-th interaction of the theory.

As we will see in the next section, all couplings of the effective theory indeed turn out to be positive. To carry out the percolation studies we will build the clusters in the Polyakov loop configurations using bond probabilities given by Eq. (6), with κ_α values taken from the effective theory. Once we have grouped all spins into clusters, we calculate the values of the following variables [13]:

- The *percolation strength* P, defined as the probability that a randomly chosen lattice site belongs to the percolating cluster. P is the *order parameter* of the percolation transition.

- The *average cluster size* S, defined as

$$S = \frac{\sum_s n_s s^2}{\sum_s n_s s} \;. \tag{7}$$

  Here, n_s is the number of clusters of size s per lattice site divided by the lattice volume, and the sums exclude the percolating cluster.

To determine the critical point of the percolation transition, we used the method suggested in [14]. For a given lattice size and a value of β we counted how many times we found a percolating cluster. This number, divided by the total number of configurations at that β, is the *percolation cumulant*. This variable is a scaling function, analogous to the Binder cumulant in continuous thermal phase transitions. Curves corresponding to different lattice sizes will thus cross at the critical point, apart from corrections to scaling.

The percolation cumulant gives not only the critical point β_c, but also the critical exponent ν. In fact, if we rewrite the percolation cumulant as a function of $(\beta - \beta_c)L^{1/\nu}$, where L is the linear spatial dimension of the lattice, we should get a universal scaling curve for all lattice sizes. We will adopt this method to determine the percolation exponent ν.

3 Numerical Results

As we are interested in the phase transition of $SU(2)$, we first focus on the derivation of the effective theory at the critical point. Simulations were done on $32^3 \times N_\tau$ lattices, with $N_\tau = 2$ and 4. For the critical coupling we took the value $\beta_c = 1.8735(4)$ for $N_\tau = 2$ as determined in [6], and $\beta_c = 2.29895(10)$ for $N_\tau = 4$ from [15]. The Monte Carlo update we used consists in alternating one heat-bath and two overrelaxation steps. We measured our observables every 60 updates, which makes the analyzed configurations basically uncorrelated; the total number of measurements was 2000. Our aim is to check whether, at $\beta = \beta_c$, we can find a projection of the theory onto the spin model defined by Eq.(2).

For $N_\tau = 2$ we considered two sets of operator containing 10 and 15 spin-spin couplings, respectively. The operators connect a point (000) to (100), (110), (111), (200), (210), (211), (220), (221), (222), (300), (310), (311), (320), (321), (322). In the case of $N_\tau = 4$ we have included further operators in our analysis, which connect the point (000) to (330), (331), (332) and (333). The final set of couplings is displayed in Table 1, where I refers to the projection with 10 (15) operators for $N_\tau = 2$ ($N_\tau = 4$) and II refers to the projection with 15 (19) operators for $N_\tau = 2$ ($N_\tau = 4$). Note that the physical length scale changes with N_τ. Two spins separated by a distance R measured in lattice units on the lattice with temporal extent N_τ are actually separated by distance r, with $rT_c = R/N_\tau$. Here $T_c = T(\beta_c)$ denotes the transition temperature of $SU(2)$ gauge theory.

From Table 1 one can see that the difference between couplings calculated for set I and set II is of the order of the statistical errors. This implies that possible truncation errors are small. Comparing our couplings for set I and $N_\tau = 2$ with the corresponding couplings obtained in [9], we see that they agree within statistical errors, although multi-spin couplings were also included in the analysis of Ref. [9]. This suggests that the multi-spin couplings are not important, for $N_\tau = 2$; in fact, most of them are compatible with zero within statistical errors. Note that because of much improved statistics, the errors on the couplings in Table 1 are by an order of magnitude smaller than in [9]. Since the error on κ_{15} is of the order of its average value, we can set $\kappa_{15} = 0$ without appreciable effects. In this way, we indeed have obtained the effective theory we were looking for, with only ferromagnetic spin-spin interactions.

Couplings	$N_\tau = 2$		$N_\tau = 4$	
	I	II	I	II
κ_1 (100)	0.13042(8)	0.13066(9)	0.08385(3)	0.08390(4)
κ_2 (110)	0.01911(1)	0.01905(3)	0.01842(4)	0.01839(5)
κ_3 (111)	0.00476(5)	0.00470(5)	0.00769(5)	0.00775(4)
κ_4 (200)	0.00794(5)	0.00801(11)	0.00702(4)	0.00697(1)
κ_5 (210)	0.00198(6)	0.00192(4)	0.00348(4)	0.00343(2)
κ_6 (211)	0.00069(3)	0.00062(8)	0.00203(5)	0.00197(1)
κ_7 (220)	0.00045(7)	0.00033(2)	0.00118(2)	0.00114(1)
κ_8 (221)	0.00013(6)	0.00007(2)	0.00082(1)	0.00083(1)
κ_9 (222)	0.00017(4)	0.00014(10)	0.00038(7)	0.00035(6)
κ_{10} (300)	0.00072(16)	0.00058(3)	0.00113(4)	0.00105(9)
κ_{11} (310)		0.00018(3)	0.00076(5)	0.00082(5)
κ_{12} (311)		0.00008(1)	0.00055(9)	0.00055(4)
κ_{13} (320)		0.00001(1)	0.00032(3)	0.00035(2)
κ_{14} (321)		0.00006(1)	0.00033(1)	0.00030(4)
κ_{15} (322)		-0.00005(6)	0.00029(3)	0.00013(4)
κ_{16} (330)				0.00020(5)
κ_{17} (331)				0.00018(3)
κ_{18} (332)				0.00017(1)
κ_{19} (333)				0.00017(4)

Table 1: Couplings of the effective theory for the Ising-projected Polyakov loop configurations of $(3+1)$-dimensional $SU(2)$. I and II denote the two sets of operator containing 10 and 15 operators for $N_\tau = 2$, and 15 and 19 operators for $N_\tau = 4$.

Fig. 1 shows a comparison between the magnetization distribution for $N_\tau = 2$ of the Polyakov loop configurations and the one of the effective theory containing 15 operators: the two histograms are very similar. The values of the average magnetization m are also in agreement: for $SU(2)$, $m = 0.091(1)$ and for the spin model, $m = 0.0923(7)$. The average magnetization m calculated from the effective theory containing 10 operators is by 10% smaller than this value. One can also clearly see differences between the magnetization distributions calculated from the Polyakov loop configurations and from the effective theory with 10 operators.

In the case of $N_\tau = 4$ we have considered two sets of spin-spin operators containing 15 and 19 couplings labeled I and II, respectively. The values of the couplings are shown in Table 1. Note that all operators are positive in this case. The average value of the magnetization m obtained from the effective theory with 19 operators is 0.121(3), which should be compared with the corresponding value calculated from SU(2) configurations, $m = 0.128(6)$. The effective theory with 15 operators gives a value which is by 20% smaller.

The couplings reported in Table 1 decrease exponentially as function of rT. The exponential fall-off is approximately described by $\sim \exp(-9rT)$ for both $N_\tau = 2$ and $N_\tau = 4$. It is easy to see that couplings of the effective spin theory at fixed rT should scale as N_τ^{-4} close to the continuum limit. A comparison of couplings for $N_\tau = 2$ and

$N_\tau = 4$ at the same value of rT_c shows that this scaling holds up to 20% scaling violation. This observation suggests that the form of the effective theory for obtained $N_\tau = 2$ and 4 will not change qualitatively by approaching the continuum limit. In particular, we expect that the effective theory will contain only ferromagnetic spin-spin coupling in the continuum limit.

To check the volume dependence of the couplings we also determined them on $16^3 \times 2$ and $16^3 \times 4$ lattices. The resulting values agree with those obtained from simulations on larger $32^3 \times 2$ and $32^3 \times 4$ lattices within statistical errors. We therefore conclude that the set of couplings we calculated is, with good approximation, also that of the effective theory in the infinite volume limit, which is what we will need for our subsequent percolation studies [1].

We have shown that the effective $Z(2)$ theory with 15 ($N_\tau = 2$) and 19 ($N_\tau = 4$) spin-spin operators gives a good description of the thermal critical behaviour of $SU(2)$ gauge theory. We now want to see if this effective theory also allows us to extend the Polyakov loop percolation description of the deconfinement transition [4, 5] from the strong coupling limit [5] to a more general situation closer to the continuum limit. To study this we generate $SU(2)$ Polyakov loop configuration on $N_\sigma^3 \times N_\tau$ lattices, reduce them to $Z(2)$ spin configurations and then define clusters with the bond weights as defined in Eq. (6). A cluster thus consists of a set of aligned spins coupled by one or more bond weights. We want to determine the percolation behaviour of these clusters.

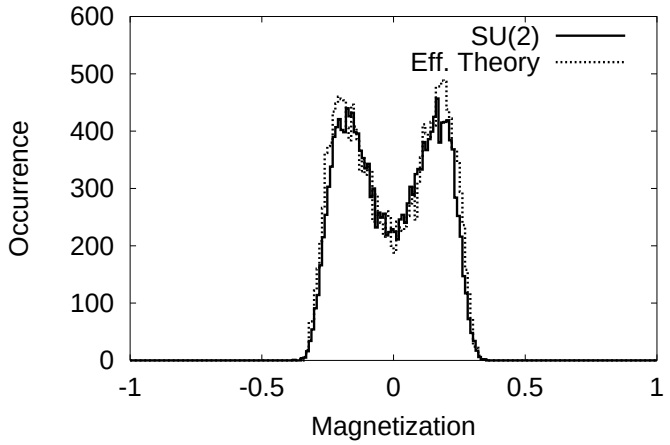

Figure 1: Comparison of the magnetization histograms derived from the Polyakov loop configurations and from the effective theory (2) defined by the couplings of Table 1.

We stress that the bond weights are temperature-dependent. Our effective theory represents a projection of $SU(2)$ for $\beta = \beta_c$. But in order to carry out our analysis, we need to evaluate the percolation variables at different values of β. Strictly speaking, for each β_i we should derive the corresponding effective theory and use the set $\{\kappa(\beta_i)\}$ to calculate the bond weights (6) at β_i. We have calculated the couplings of the effective theory for several values of β in the interval [1.873,1.883] for $16^3 \times 2$ lattice. It turned

[1]The bond weights are determined by the temperature. Once we fix the temperature, for different lattice sizes we must use the same set of values for the bond weights.

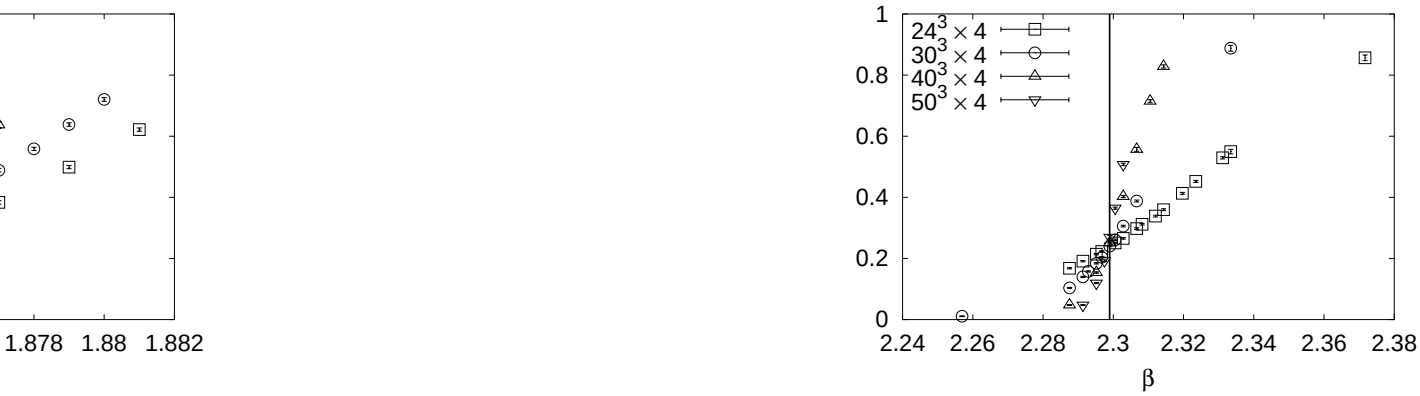

Figure 2: Percolation cumulants near the critical point for different spatial volume for $N_\tau = 2$ (left) and for $N_\tau = 4$ (right). The solid vertical lines indicate the value of the thermal threshold. For $N_\tau = 2$ we also indicate the uncertainty on the threshold by two dashed lines. For $N_\tau = 4$ the errors on the critical coupling are too small to be visible in the Figure.

out that the variation of the couplings due to the variation of β in this small interval is of the order of the statistical errors. Because of that, at each β, we shall use the same set of bond probabilities, determined by the couplings of Table 1.

For the analysis of the percolation transition we have used four different lattices of size $N_\sigma^3 \times N_\tau$, with $N_\sigma = 24, 30, 40, 50$ and $N_\tau = 2, 4$. Fig. 2 shows the behavior of the percolation cumulant as a function of β. In both cases the curves cross remarkably well at the same point, in excellent agreement with the thermal thresholds, indicated by the solid line.

Next, we performed further simulations near criticality in order to study the critical exponents and to determine more precisely the value of the geometrical threshold. From standard finite size scaling fits of the average cluster size S at different β values we determine the value β_c at which the scaling fit gives the best χ^2; the slope of the straight line of the log-log plot of S at β_c gives γ/ν. The errors on β_c and on the ratio of critical exponents are calculated by determining the β-range containing β_c such that for each β one still gets a good χ^2 for the scaling fit. Unfortunately we could not determine β/ν, because of strong fluctuations of the percolation strength P around β_c. The value of P at criticality is in general quite small, and it is more strongly influenced by the approximations involved in our approach. Furthermore, we determine the exponent ν from the finite size scaling of the pseudo-critical points, i. e. the positions of the peak of S for the different lattices we considered.

The percolation threshold and the critical exponents are summarized in Table 2 for both $N_\tau = 2$ and 4. There we also give the critical exponents of the thermal $SU(2)$ transition and those corresponding to the 3D Ising model. As one can see from Table 2, good agreement between the percolation exponents and the 3D Ising exponents is found. In Fig. 3 we show the scaling of the percolation cumulant with the Ising exponent and with the 3-dimensional random percolation exponent $\nu = 0.88$ for the case of $N_\tau = 2$. The figure shows that scaling is quite consistent with the Ising exponent while the random percolation exponent is ruled out by the data. The situation is the same for $N_\tau = 4$ as one can see in Fig. 4.

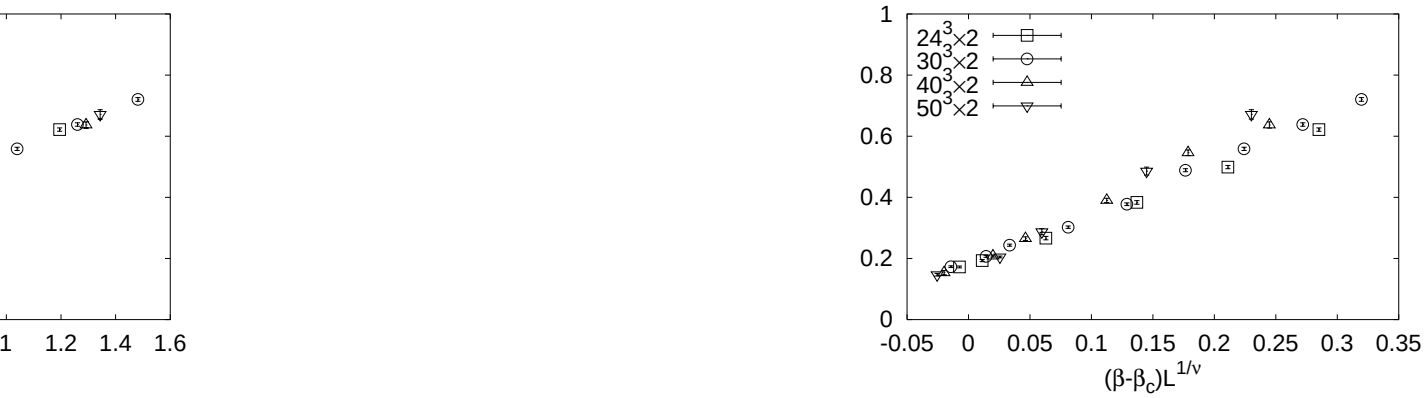

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Figure 3: Rescaling of the percolation cumulant curves shown in Fig. 2 for $N_\tau = 2$ with the 3-dimensional Ising exponent $\nu = 0.6294$ (left) and with the 3-dimensional random percolation exponent $\nu = 0.88$ (right), using $\beta_c = 1.8734$ for the critical coupling.

	Critical point		γ/ν		ν	
	$N_\tau = 2$	$N_\tau = 4$	$N_\tau = 2$	$N_\tau = 4$	$N_\tau = 2$	$N_\tau = 4$
Percolation	1.8734(2)	2.2991(2)	1.977(17)	1.979(16)	0.628(11)	0.629(11)
Thermal	1.8735(4)	2.29895(10)	1.953(9)	1.944(13)	0.630(10)	0.630(11)
3D Ising			1.967(7)		0.6294(10)	

Table 2: Percolation critical indices for $(3+1)$-dimensional $SU(2)$. For comparison we also list the thermal results [6] and the 3D Ising values [16].

4 Conclusions

We have shown that in the vicinity of the critical point $SU(2)$ gauge theory can be well approximated by a simple $Z(2)$ model with ferromagnetic spin-spin interactions alone. We have tested the accuracy of the projection for $N_\tau = 2$ and 4. In both cases we have found that there is basically no trace of frustration. The percolation picture of the effective model is suitable to describe the deconfining transition of $SU(2)$ in purely geometrical terms.

In principle the present approach can be extended also to larger values of N_τ (smaller lattice spacing). We expect that for larger N_τ more operators should be included in the effective action. This can be avoided by first performing a block spin transformation of the Polyakov loop configurations and then doing the whole analysis for the blocked configurations. We have shown that the couplings of the effective theory decrease exponentially in strength as function of rT_c and their relative magnitude scales with N_τ^{-4}. This suggests that the effective theory remains well defined in the continuum limit and will stay ferromagnetic and short ranged. Therefore we do not expect that the present picture will change qualitatively for larger N_τ.

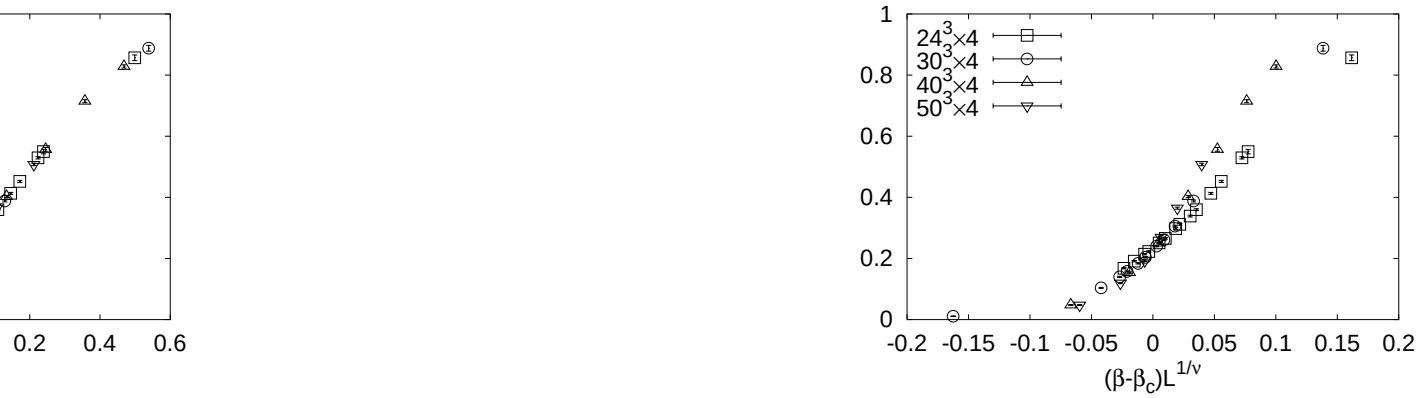

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Figure 4: Rescaling of the percolation cumulant curves shown in Fig. 2 for $N_\tau = 4$ with the 3-dimensional Ising exponent $\nu = 0.6294$ (left) and with the 3-dimensional random percolation exponent $\nu = 0.88$ (right), using $\beta_c = 2.2991$ for the critical coupling.

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